Crash-Severity Modeling

Part 2

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Introduction

- The crash severity model is to use statistical methods for identifying factors that are significantly associated with the consequence of a traffic crash, and their relationships.
- The response variable is the person who sustains the most severe injury in a crash in the KABCO scale (i.e., killed, incapacitating injury, non-incapacitating injury, possible injury, and no injury).
- A variety of models have been developed to account for data issues and methodological limitations.
- The common modeling approaches include logistic, probit and their variations.
- The impact of a factor on the injury severity levels can be estimated through its marginal effect or odds ratio.

Objectives

- Learn the characteristics of crash injury severity data.
- Gain the knowledge about data limitations and modeling challenges.
- Understand the assumptions, property and limitations of models for crash injury severity levels.
- Develop crash injury severity models and perform analysis.
- Interpret the modeling results.

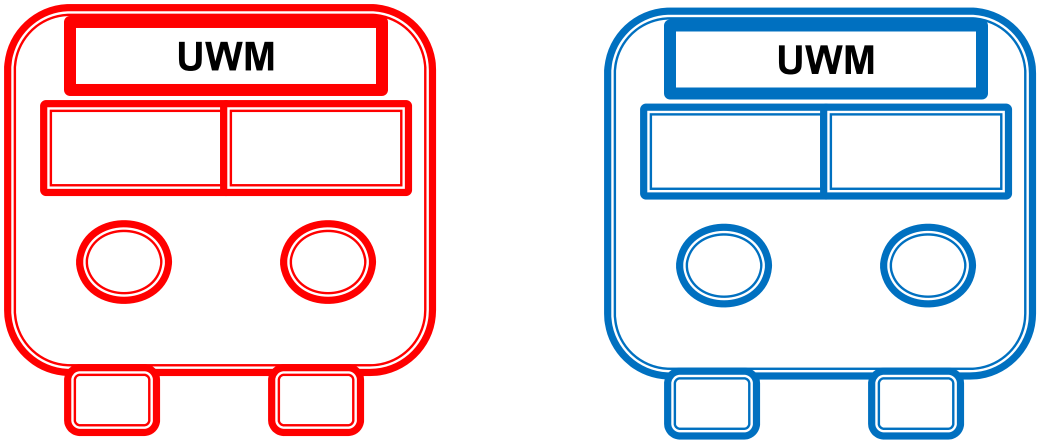
Recap

- Injury severity has a finite number of outcomes that are categorized on the KABCO scale
- KABCO scale and AIS scale
- Imbalanced observations across at different scales
- Reporting inconsistency and underreporting
- Non-ordinal or ordered probabilistic if an ordinal structure for the response variable is assumed; fixed or random parameter models, depending on the assumption for model coefficient estimation
- Sample size for injury severity models
- Utility, utility function and random utility function
- Probit model and logistic regression model: model assumption, model estimation and model results interpretation (e.g., probability, odds, odds ratio, marginal effect, elasticity)
- Model variations are available if restrictions such as irrelevant and independent alternatives (IIA), proportional odds are relaxed

The Nested Logit Model (Generalized Extreme Value Models)

Blue Bus, Red Bus

Can we increase the bus ridership by painting the bus with a different color?



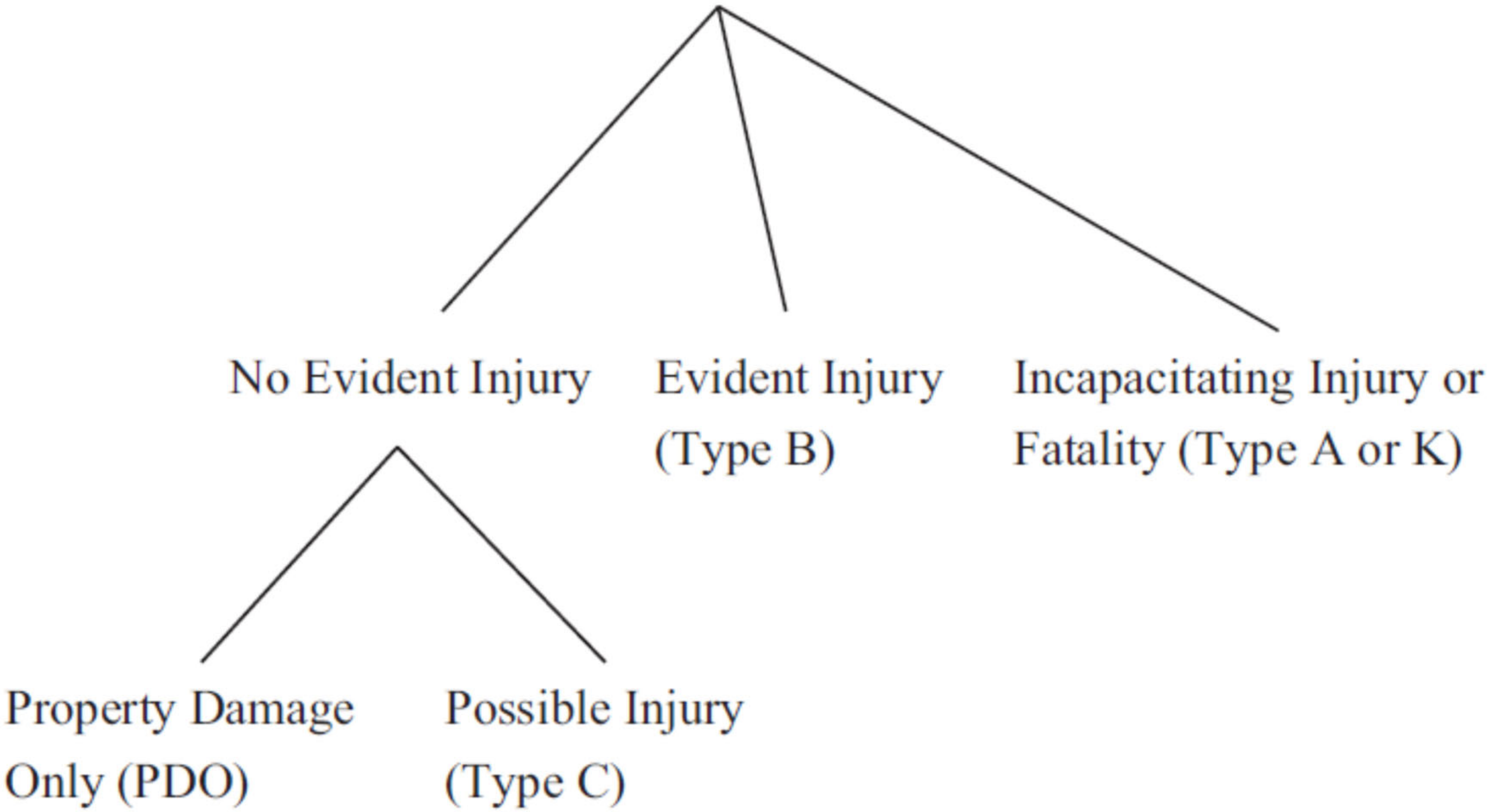
Issues with Logit Model: Red Bus Blue Bus

- Clearly, the bus share should not have been changed. What is wrong?
- The problem is with the underlying assumption in the logit model. The logit model requires that alternatives be independent (i.e. ε_{red} and ε_{blue} be independent). This is not the case in this example.
- Obviously, the errors of the perceived utility from alternative of the red bus is dependent on the error from the blue bus, and vice versa. This does not justify the use of logit model.
- Note: when the alternatives are distinctly different and independent, the logit model shall work well.

Nested Logit Model

- To overcome the IIA problem, the idea behind a nested logit model is to group alternate outcomes suspected of sharing unobserved effects into nests (this sharing sets up the disturbance term correlation that violates the derivation assumption).
- Because the outcome probabilities are determined by differences in the functions determining these probabilities (both observed and unobserved), shared unobserved effects will cancel out in each nest providing that all alternatives in the nest share the same unobserved effects.

Nested Logit Model Structure



No Evident Injury

Evident Injury (Type B)

Incapacitating Injury or Fatality (Type A or K)

Property Damage Only (PDO)

Possible Injury (Type C)

FIGURE 4.3 Nested structure of accident severities.

A General Formation

- Mathematically, McFadden (1981) has shown the GEV disturbance assumption leads to the following model structure for observation *n* choosing outcome *i*

$$P_n(i) = \frac{\exp[\boldsymbol{\beta}_i \mathbf{X}_{in} + \phi_i LS_{in}]}{\sum_{\forall I} \exp[\boldsymbol{\beta}_I \mathbf{X}_{In} + \phi_I LS_{In}]} \qquad (4.8a)$$

$$P_n(j|i) = \frac{\exp[\boldsymbol{\beta}_{j|i} \mathbf{X}_n]}{\sum_{\forall J} \exp[\boldsymbol{\beta}_{J|i} \mathbf{X}_{Jn}]} \qquad (4.8b)$$

$$LS_{in} = LN[\sum_{\forall J} \exp(\boldsymbol{\beta}_{J|i} \mathbf{X}_{Jn})], \qquad (4.8c)$$

Where

$P_n(i)$ is the unconditional probability of observation n having discrete outcome i,

X's are vectors of characteristics that determine the probability of discrete outcomes,

β's are vectors of estimable parameters,

$P_n(j|i)$ is the probability of observation n having discrete outcome j conditioned on the outcome being in outcome category i,

J is the conditional set of outcomes (conditioned on i), I is the unconditional set of outcome categories, LS_{in} is the inclusive value (logsum), and ϕ_i is an estimable parameter.

ϕ

- In order to be consistent with McFadden’s generalized extreme value derivation of the model, the parameter estimate for ϕ_i in the nested logit model must be between zero and one.
- If ϕ_i equals to one or is not significantly different from one, there is no correlation between the severity levels in the nest, meaning the model reduces to the multinomial logit model.
- If ϕ_i equals to zero, a perfect correlation is implied among the severity levels in the nest, indicating a deterministic process by which crashes result in particular severity levels.
- The t test can be used to test if ϕ_i is significantly different from 1. Because ϕ_i is less than or equal to one, this is a one-tailed t test (half of the two-tailed t-test).
- It is important to note that the typical t-test implemented in many commercial software packages are against zero instead of one. Thus, the t value must be calculated manually. The IIA assumption for a MNL model can also be tested with the Hausman-McFadden (1984) test which has been widely implemented in commercial statistical software.

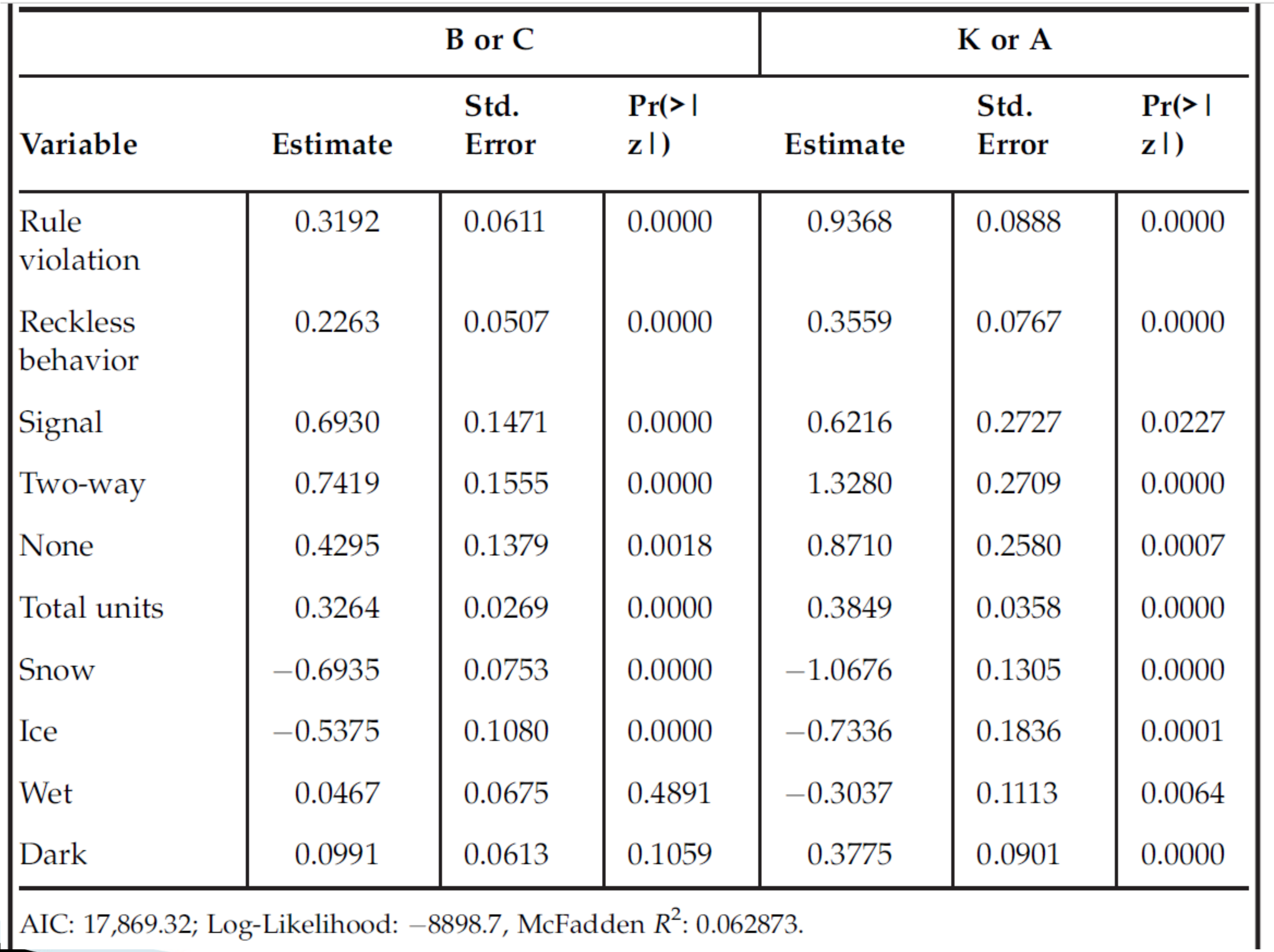
Estimation of a Nested Model

- Done in a sequential fashion.
  - Estimate the conditional model using only the observations in the sample that are observed having discrete outcomes *J*.
  - Once these estimation results are obtained, the logsum is calculated (this is the denominator of one or more of the conditional models) for all observations, both those selecting *J* and those not.
- The full information maximum likelihood (FIML) estimation (Greene, W., Econometric Analysis, 2000).

NL Model Coefficient Estimates

Exercise 4.2: Coefficient Estimates for NL*

*: partial results shown here

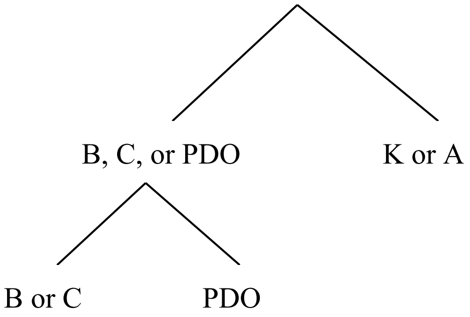


	B or C			K or A		
Variable	Estimate	Std. Error	Pr(>|z|)	Estimate	Std. Error	Pr(>|z|)
Rule violation	0.3192	0.0611	0.0000	0.9368	0.0888	0.0000
Reckless behavior	0.2263	0.0507	0.0000	0.3559	0.0767	0.0000
Signal	0.6930	0.1471	0.0000	0.6216	0.2727	0.0227
Two-way	0.7419	0.1555	0.0000	1.3280	0.2709	0.0000
None	0.4295	0.1379	0.0018	0.8710	0.2580	0.0007
Total units	0.3264	0.0269	0.0000	0.3849	0.0358	0.0000
Snow	−0.6935	0.0753	0.0000	−1.0676	0.1305	0.0000
Ice	−0.5375	0.1080	0.0000	−0.7336	0.1836	0.0001
Wet	0.0467	0.0675	0.4891	−0.3037	0.1113	0.0064
Dark	0.0991	0.0613	0.1059	0.3775	0.0901	0.0000

AIC: 17,869.32; Log-Likelihood: −8898.7, McFadden R^2: 0.062873.

Solution

1. Establish the nested structure of crash severities. :



2. Determine the functional form based on Eq. 4.8 (a), (b) and (c). For example, $P_n(j|i)$ is the probability of crash n having injury outcome B or C conditioned on the injury outcome being in category not a K or A injury. I is the unconditional set of outcome categories (for example, the upper three branches in the figure: no K/A injury and K/A injury). LSni is the inclusive value (logsum).

3. Estimate the coefficients using the R "mlogit" package:

```
nested_logit <- mlogit(INJSVR ~ 0|YOUNG + OLD + FEMALE + ALCFLAG + DRUGFLAG + SAFETY +
DRVRPC_SPD + DRVRPC_RULEVIO + DRVRPC_RECK + TRFCONT_SIGNAL + TRFCONT_2WAY +
TRFCONT_NONE + TOTUNIT + ROADCOND_SNOW + ROADCOND_ICE + ROADCOND_WET +
LGTCOND_DARK, data = crash_mnl, nests = list(KA = c("3"), non_KA = c("1", "2")), un.nest.el = TRUE).
```

4. summarize your findings. The AIC value of the NL model (17871.11) is greater than that of MNL model (17869.32), indicating inferior performance. The inclusive value is 0.7161 and its t-value is -0.474. Apparently, the log-sum coefficient is not significantly different from 1. When the inclusive value is equal to one or not significantly different from 1, there is no correlation between the severity levels in the nest. We can conclude that for this dataset, the MNL model is more appropriate.

Mixed Logit: Motivation

- More aggregate – cannot include specific accident characteristics (driver characteristics, vehicle characteristics, restraint usage, alcohol consumption, etc.).
- Without detailed accident information, the approach potentially introduces a heterogeneity problem.
- Heterogeneity could result in varying effects of X that could be captured with random parameters.
- Mixed logit may be appropriate.
  - Relaxes possible IIA problems with a more general error-term structure.
  - Can test a variety of distribution options for β .
  - Estimated with simulation based maximum likelihood.

Mixed Logit Model

Similar to the random parameter model for crash-frequency. This means that the coefficients are allowed to vary across observations.

$$P_{ni}(i) = \int \frac{\exp(\mathbf{x}'_{ni}\boldsymbol{\beta}_i)}{\sum_J \exp(\mathbf{x}'_{nJ}\boldsymbol{\beta}_J)} f(\boldsymbol{\beta}|\boldsymbol{\phi})\,d\boldsymbol{\beta} \tag{4.9}$$

where $f(\boldsymbol{\beta}|\boldsymbol{\varphi})$ is a density function of $\boldsymbol{\beta}$ and $\boldsymbol{\varphi}$ is a vector of parameters which specify the density function, with all other terms as previously defined.

In a statistics term, the weighted average of several functions is called a mixed function, and the density that provides the weights is called the mixing distribution. Mixed logit is a mixture of the standard logit function evaluated at different $\boldsymbol{\beta}$ with $f(\boldsymbol{\beta})$ being the mixing distribution.

Model Specifications

- The injury severity level probability is a mixture of logits. When all parameters β are fixed, the model reduces to the multinomial logit model.
- When β is allowed to vary, the model is not in a closed form, and the probability of crash observation n having a particular injury outcome i can be calculated through integration.
- Simulation-based maximum likelihood methods such as Halton draws are usually used.
- The choice of the density function of β depends on the nature of the coefficient and the statistical goodness of fit.
  - The lognormal distribution is useful when the coefficient is known to have the same sign for each observation.
  - Triangular and uniform distributions have the advantage of being bounded on both sides.
  - Furthermore, triangular assumes that the probability increases linearly from the beginning to the mid-range and then decreases linearly to the end.
  - A uniform distribution assumes the same probability for any value within the range.

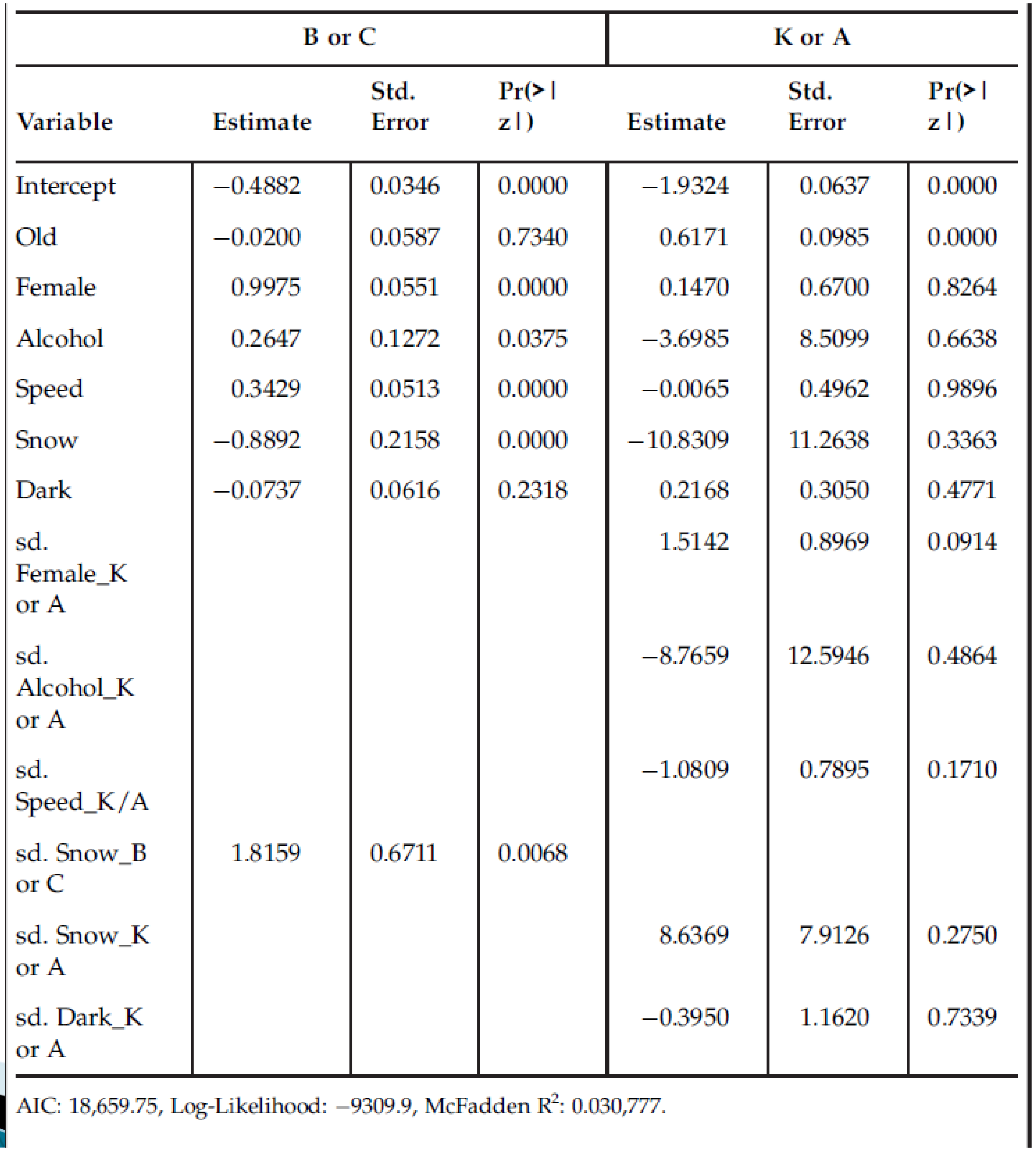
Random Coefficient or Error Components?

- Random coefficient: $U_{nj} = \beta_n' x_{nj} + \varepsilon_{nj}$ where β_n can be decomposed into mean α and deviations μ_n such as $(\alpha' x_{nj} + \mu_n' x_{nj})$ and ε_{nj} is a random term that is iid extreme value.
- Error components: $U_{nj} = \alpha' x_{nj} + \mu_n' z_{nj} + \varepsilon_{nj}$ where x_{nj} and z_{nj} are vectors of observable variables relating to alternative j. α is a vector of fixed parameters and μ is random with zero mean, and ε_{nj} is iid extreme value. So, the random portion of utility is $(\mu_n' z_{nj} + \varepsilon_{nj})$ which can be correlated over alternatives depending on z.
- Error-component and random-coefficient specifications are formally equivalent; but a researcher thinks about the model affects the specification of the mixed logit.
- It is important to know that the mixing distribution, whether driven by random parameters or by error components, captures variance and correlations in unobserved factors. But there is a limit on how much one can learn about things that are not seen.

ML Model Coefficient Estimates

Exercise 4.3: Coefficient Estimates for ML*

*: partial results shown here



	B or C			K or A		
Variable	Estimate	Std. Error	Pr(>|z|)	Estimate	Std. Error	Pr(>|z|)
Intercept	−0.4882	0.0346	0.0000	−1.9324	0.0637	0.0000
Old	−0.0200	0.0587	0.7340	0.6171	0.0985	0.0000
Female	0.9975	0.0551	0.0000	0.1470	0.6700	0.8264
Alcohol	0.2647	0.1272	0.0375	−3.6985	8.5099	0.6638
Speed	0.3429	0.0513	0.0000	−0.0065	0.4962	0.9896
Snow	−0.8892	0.2158	0.0000	−10.8309	11.2638	0.3363
Dark	−0.0737	0.0616	0.2318	0.2168	0.3050	0.4771
sd. Female_K or A				1.5142	0.8969	0.0914
sd. Alcohol_K or A				−8.7659	12.5946	0.4864
sd. Speed_K/A				−1.0809	0.7895	0.1710
sd. Snow_B or C	1.8159	0.6711	0.0068			
sd. Snow_K or A				8.6369	7.9126	0.2750
sd. Dark_K or A				−0.3950	1.1620	0.7339

AIC: 18,659.75, Log-Likelihood: −9309.9, McFadden R^2: 0.030,777.

Solution

1. **Determine the density function in the R “mlogit” package**, random parameter object “rpar” contains all the relevant information about the distribution of random parameters. Currently, the normal ("n"), log-normal ("ln"), zero-censored normal ("cn"), uniform ("u") and triangular ("t") distributions are available. For illustration, normal distribution is chosen as the density function of random parameter β.
2. **Estimate the coefficients using the R “mlogit” package:**

```
crash_data_mixed <- mlogit.data(data_mixed_ch4, shape = "long", choice = "INJSVR", chid.var = "ID",
alt.var = "OUTCOME")

mixed_logit <- mlogit(INJSVR ~ OLD_2 + OLD_3 + FEMALE_2 + FEMALE_3 + ALCFLAG_2 + ALCFLAG_3
+ DRVRPC_SPD_2 + DRVRPC_SPD_3 + ROADCOND_SNOW_2 + ROADCOND_SNOW_3 +
LGTCOND_DARK_2 + LGTCOND_DARK_3, data = crash_data_mixed, rpar = c(FEMALE_2 = 'n’,
FEMALE_3 = 'n', ALCFLAG_3 = 'n', DRVRPC_SPD_3  = 'n', ROADCOND_SNOW_2 = 'n’,
ROADCOND_SNOW_3 = 'n', LGTCOND_DARK_3 = 'n'),panel = FALSE, correlation = FALSE, R = 100,
halton = NA).
```

Note: rpar argument names random coefficients (‘n’ for a normal distribution); halton=NA means default halton draws are applied. (if interested, read “Halton Sequences for Mixed Logit” by Kenneth Train at https://eml.berkeley.edu/wp/train0899.pdf)

3. **Summarize the findings**. The ML model can account for the data heterogeneity by treating coefficients as random variables. the snowy surface parameter for truck K or A injuries is fixed (-10.830); and for severity B or C, it is normally distributed with a mean of -0.8892 and a standard deviation of 1.8159, meaning that 31% of truck crashes occurring on snowy pavement have an increased possibility of B or C injuries. It is plausible that people often drive more slowly and cautiously on snowy roads but that the slick conditions still have a tendency to cause accidents.

- In Milton et al. (2008), the application of the mixed logit model (also called the random parameters logit model) is undertaken by considering injury-severity proportions for individual roadway segments.
- For all of the random parameters, the normal distribution was found to provide the best statistical fit (among normal, lognormal, triangular and uniform).

“the constant for the property-damage only proportion is normally distributed with mean −0.355 and standard deviation 1.776. ….This variability is likely capturing the unobserved heterogeneity in the roadway segments that could include factors such as visual noise and other physical and environmental factors. ….The average daily traffic (ADT) per lane is normally distributed with a mean 0.0403 and standard deviation 0.515. … 46.9% of the distribution is less than 0 and 53.1% is greater than 0…. a complex interaction among traffic volume, driver behavior and accident-injury severity.”

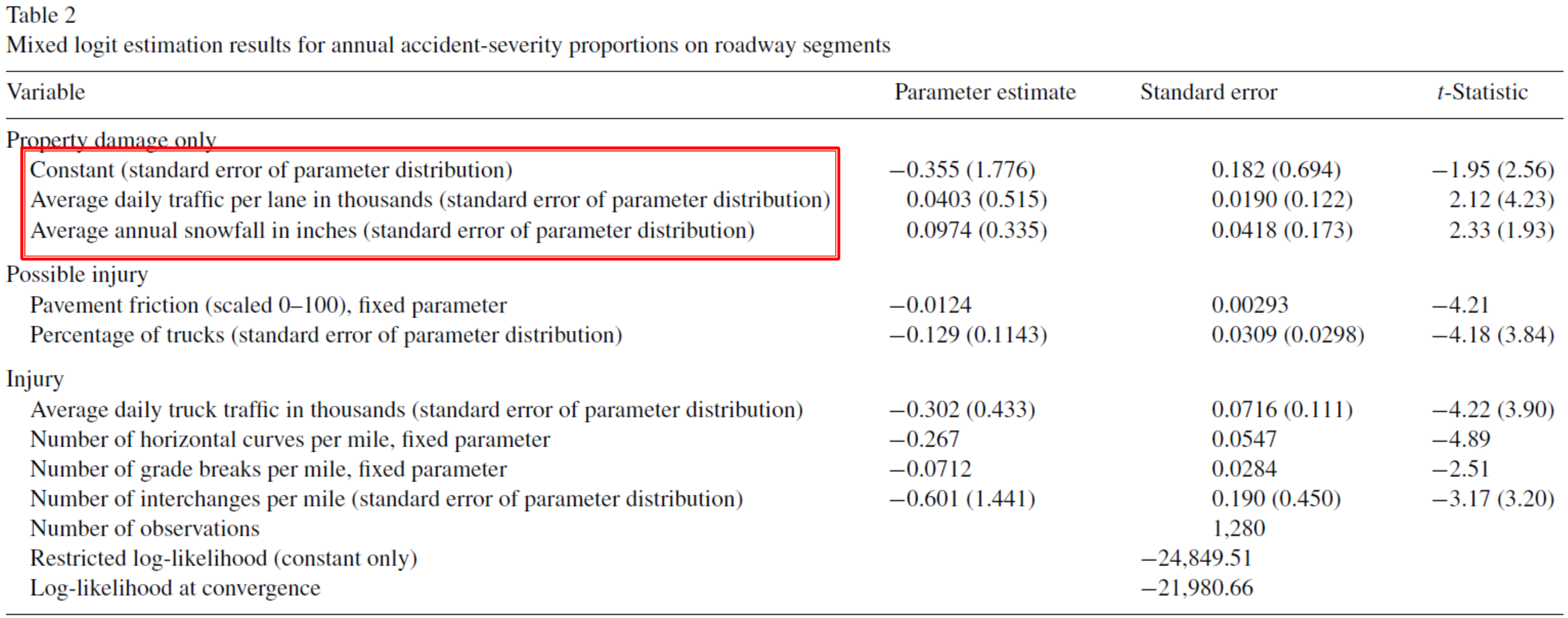


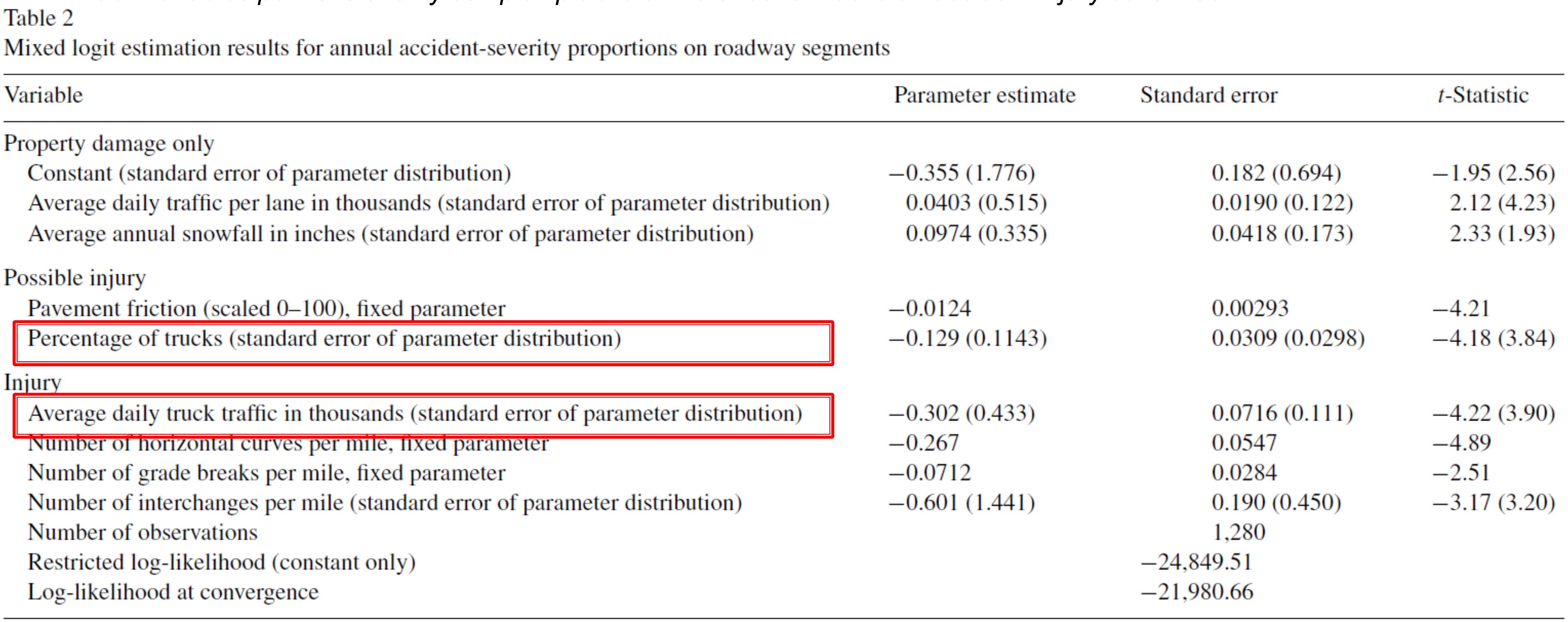
Table 2
Mixed logit estimation results for annual accident-severity proportions on roadway segments

Variable	Parameter estimate	Standard error	*t*-Statistic
Property damage only			
Constant (standard error of parameter distribution)	−0.355 (1.776)	0.182 (0.694)	−1.95 (2.56)
Average daily traffic per lane in thousands (standard error of parameter distribution)	0.0403 (0.515)	0.0190 (0.122)	2.12 (4.23)
Average annual snowfall in inches (standard error of parameter distribution)	0.0974 (0.335)	0.0418 (0.173)	2.33 (1.93)
Possible injury			
Pavement friction (scaled 0–100), fixed parameter	−0.0124	0.00293	−4.21
Percentage of trucks (standard error of parameter distribution)	−0.129 (0.1143)	0.0309 (0.0298)	−4.18 (3.84)
Injury			
Average daily truck traffic in thousands (standard error of parameter distribution)	−0.302 (0.433)	0.0716 (0.111)	−4.22 (3.90)
Number of horizontal curves per mile, fixed parameter	−0.267	0.0547	−4.89
Number of grade breaks per mile, fixed parameter	−0.0712	0.0284	−2.51
Number of interchanges per mile (standard error of parameter distribution)	−0.601 (1.441)	0.190 (0.450)	−3.17 (3.20)
Number of observations		1,280	
Restricted log-likelihood (constant only)		−24,849.51	
Log-likelihood at convergence		−21,980.66	

Milton, J. C., Shankar, V. N., & Mannering, F. L. (2008). Highway accident severities and the mixed logit model: an exploratory empirical analysis. Accident Analysis & Prevention, 40(1), 260-266.

- In Milton et al. (2008), the application of the mixed logit model (also called the random parameters logit model) is undertaken by considering injury-severity proportions for individual roadway segments.

"*The percentage of trucks…had a mean of−0.129 and standard deviation 0.1143, being less than 0 for 87.1% of the roadway segments and greater than 0 for 12.9% of the segments…in a small proportion of roadway segments, the truck percentage increases the proportion of possible injury accidents, while in a majority of roadway segments, the proportion tends to decrease. Note that this variable implies that for 87.1% of roadway segments increasing truck percentages make the severity proportions more likely to be minor (property damage only) or major (injury)... 75.2% of the roadway segments negative values (an increasing number of trucks decreases the likelihood of accidents resulting in injury) and 24.8% positive values (an increasing number of trucks increases the likelihood of accidents resulting in injury).The net effect of these two truck variables points to a fairly complex picture of the effect of trucks on accident-injury severities.*"



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Ordered Response Models

Modeling Crash Severity as an Ordered Discrete Outcome

- The primary rationale for using ordered discrete choice models for modeling crash severity is that there is an intrinsic order among injury severities, with fatality being the highest order and property damage being the lowest. Including the ordinal nature of the data in the statistical model defends the data integrity and preserves the information.
- Second, the consideration of ordered response models avoids the undesirable properties of the multinomial model such as the IIA in the case of a multinomial logit model or a lack of closed-form likelihood in the case of a multinomial probit model.
- Third, ignoring the ordinality of the variable may cause a lack of efficiency (i.e., more parameters may be estimated than are necessary if the order is ignored).
- Although there are many positives to the ordered model, the disadvantage is that imposing restrictions on the data may not be appropriate despite the appearance of a rank. Therefore, it is important to test the validity of the ordered restriction.

Ordered Logit/Probit Model Structure

- Ordered probability models are derived by defining an unobserved variable Z that is used as a basis for modeling the ordinal ranking of data.
- Observed ordinal data, *y*, for each observation are defined as,
- A high indexing of z_n is expected to result in a high level of observed injury y_n in the case of a crash. The observed discrete injury severity variable y_n is stratified by thresholds as follows:

$$y_n = \begin{cases} 1, if\ z_n \le \mu_1 \left(\text{PDO or no injury}\right) \\ 2, if\ \mu_1 < z_n \le \mu_2 \left(\text{injury C}\right) \\ 3, if\ \mu_2 < z_n \le \mu_3 \left(\text{injury B}\right) \\ 4, if\ \mu_3 < z_n \le \mu_4 \left(\text{injury A}\right) \\ 5, if\ \mu_4 < z_n \left(\text{K or fatal injury}\right) \end{cases} \qquad (4.11)$$

Where μ's are estimable parameters (referred to as thresholds) that define y, and y corresponds to integer ordering, and I is the highest integer ordered response.

Ordered Logit/Probit

The ordinal logit/probit model applies a latent continuous variable, z_n, as a basis for modeling the ordinal nature of crash severity data, and z_n is specified as a linear function of $\boldsymbol{X_n}$:

$$z_n = \boldsymbol{\beta}'\boldsymbol{X_n} + \varepsilon_n \qquad (4.10)$$

Where $\boldsymbol{X_n}$ is a vector of explanatory variables determining the discrete ordering (i.e., injury severity) for *n* th crash observation, $\boldsymbol{\beta}$ is a vector of estimable parameters, and ε_n is an error term that accounts for unobserved factors influencing injury severity.

Ordered Probit Model

ε is assumed to be **normally distributed** across observations with N(0,1), resulting in an **ordered probit model**

$$P_n(1) = \Pr(y_n = 1) = \Pr(z_n \le \mu_1) = \Pr(\mathbf{x}_n'\boldsymbol{\beta} + \varepsilon_n \le \mu_1) = \Pr(\varepsilon_n \le \mu_1 - \mathbf{x}_n'\boldsymbol{\beta})$$

$$= \Phi(\mu_1 - \mathbf{x}_n'\boldsymbol{\beta})$$

$$P_n(2) = \Pr(y_n = 2) = \Pr(\mu_1 < z_n \le \mu_2) = \Pr(\mu_1 < \mathbf{x}_n'\boldsymbol{\beta} + \varepsilon_n \le \mu_1)$$

$$= \Pr(\varepsilon_n \le \mu_2 - \mathbf{x}_n'\boldsymbol{\beta}) - \Pr(\varepsilon_n \le \mu_1 - \mathbf{x}_n'\boldsymbol{\beta})$$

$$= \Phi(\mu_2 - \mathbf{x}_n'\boldsymbol{\beta}) - \Phi(\mu_1 - \mathbf{x}_n'\boldsymbol{\beta}) \qquad (4.12)$$

$$\vdots$$

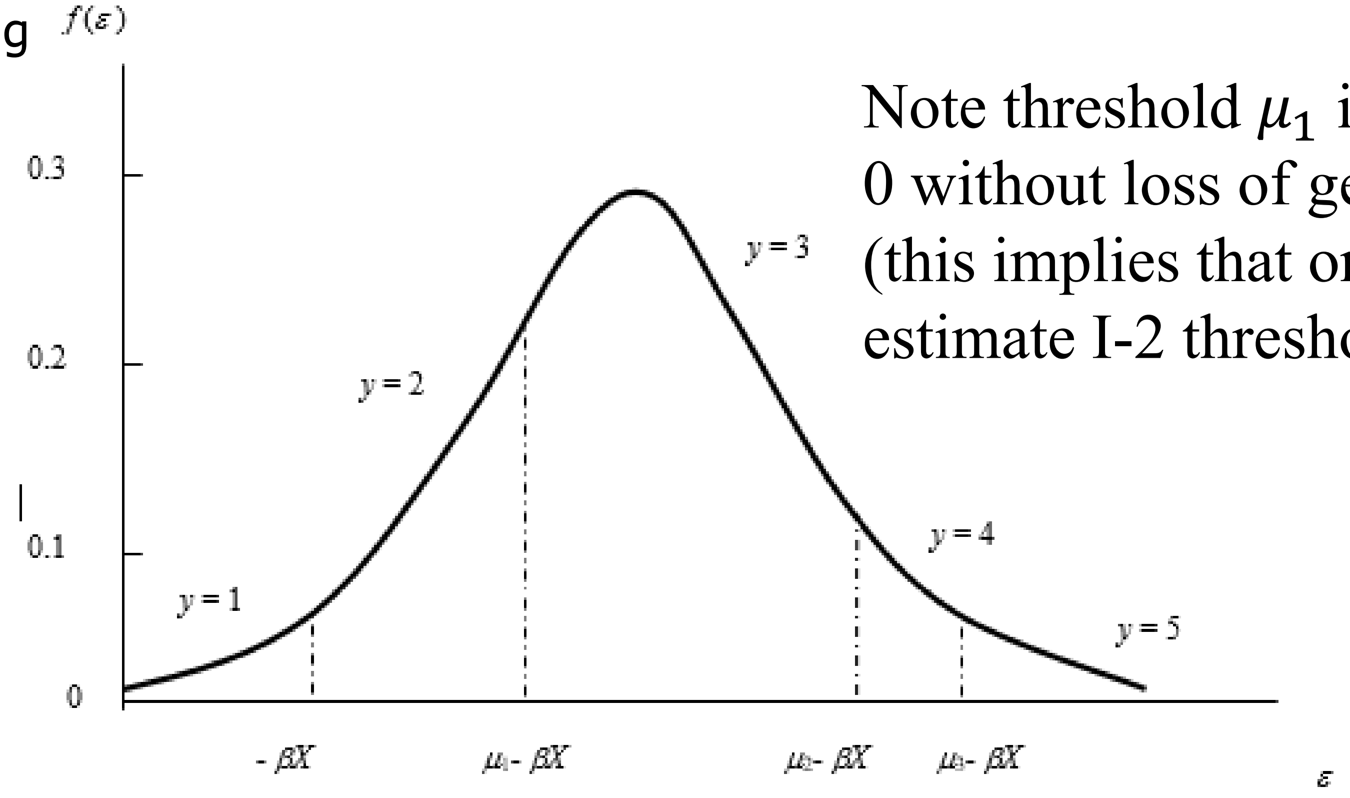
$$P_n(i+1) = \Phi(\mu_{i+1} - \mathbf{x}_n'\boldsymbol{\beta}) - \Phi(\mu_i - \mathbf{x}_n'\boldsymbol{\beta})$$

$$\vdots$$

$$P_n(I) = Pr(y_n = I) = Pr(z_n > \mu_{I-1}) = 1 - \Phi(\mu_{I-1} - \boldsymbol{x}_n'\boldsymbol{\beta})$$

Ordered Probit Model (cont'd)

If ε is assumed to be **normally distributed** across observations with N(0,1), an **ordered probit model** results with the ordered selection probabilities being



Note threshold μ_1 is set equal to 0 without loss of generality (this implies that one need only estimate I-2 thresholds.

Illustration of an ordered probability model with μ_1=0

Limitation

- The difficulty arises because the areas between the shifted thresholds may yield increasing or decreasing probabilities after shifts to the left or right, especially for the intermediate categories (i.e., y=2, y=3, and y=4).
- The change depends on the location of the thresholds.
- A trade-off is inherently being made between recognizing the ordering of responses and losing the flexibility in specification offered by unordered probability models.

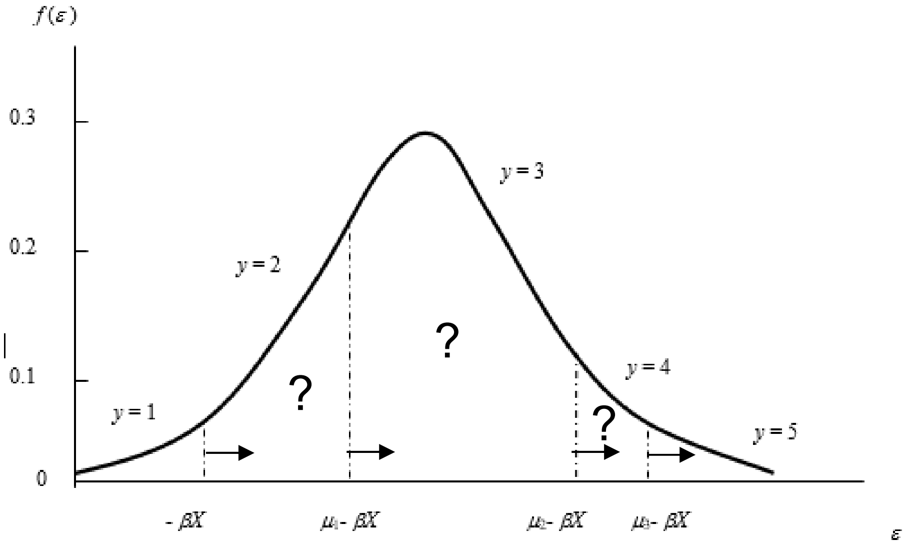


Illustration of an ordered probability model with $\mu_0 = 0$.

Estimation

$P(u = i) = \Phi(\mu_i - \beta X) - \Phi(\mu_{i-1} - \beta X)$

Where μ_i and μ_{i-1} represent the upper and lower thresholds for outcome i.

The likelihood function is:

$$L(u|\beta,\mu) = \prod_{n=1}^{N}\prod_{i=1}^{I}[\Phi(\mu_i - \beta X_n) - \ \Phi(\mu_{i-1} - \beta X_n)]^{\delta_{in}}$$

$$LL(u|\beta,\mu) = \sum_{n}\sum_{i}\delta_{in}LN[\Phi(\mu_i - \beta X_n) - \ \Phi(\mu_{i-1} - \beta X_n)]$$

where $\delta_{in} = 1$ if the observed discrete outcome for observation n is i, and zero otherwise.

Maximize the LL is subject to the constraint $0 \leq \mu_1 \leq \mu_2 \ldots \leq \mu_{I-2}$

Order Logit Model

- Ordered logit can also be conceptualized as a latent variable model.
- Let Z be a continuous random variable that depends on a set of explanatory variables X, $Z = \beta X + \varepsilon$, that is used as a basis for modeling the ordinal ranking of data.
- If we assume that ε follows **a standard logistic distribution**, it follows the cumulative logit, also known as ordered or ordinal logit model.

- Recall 4.11, $$y_n = \begin{cases} 1, if\ z_n \le \mu_1 \left(\text{PDO or no injury}\right) \\ 2, if\ \mu_1 < z_n \le \mu_2 \left(\text{injury C}\right) \\ 3, if\ \mu_2 < z_n \le \mu_3 \left(\text{injury B}\right) \\ 4, if\ \mu_3 < z_n \le \mu_4 \left(\text{injury A}\right) \\ 5, if\ \mu_4 < z_n \left(\text{K or fatal injury}\right) \end{cases}$$

$$\begin{aligned} \Pr\left(y_n > i\right) &= \Pr\left(Z_n > \mu_i\right) = \Pr\left(\varepsilon_n > \mu_i - \mathbf{x}_n'\boldsymbol{\beta}\right) \\ &= \frac{1}{1+\exp\left(\mu_i - \mathbf{x}_n'\boldsymbol{\beta}\right)} = \frac{\exp\left(\mathbf{x}_n'\boldsymbol{\beta} - \mu_i\right)}{1+\exp\left(\mathbf{x}_n'\boldsymbol{\beta} - \mu_i\right)} \end{aligned} \qquad (4.15)$$

Where ε_n follows logistic distribution whose CDF is: $F\left(\varepsilon_n\right) = \dfrac{\exp\left(\varepsilon_n\right)}{1+\exp\left(\varepsilon_n\right)}$

So, $Pr(\varepsilon_n > \mu_i - \boldsymbol{x}_n'\boldsymbol{\beta}) = 1 - F(\varepsilon_n) = \frac{1}{1+exp(\mu_i - \boldsymbol{x}_n'\boldsymbol{\beta})}$.

- Model can be specified as below

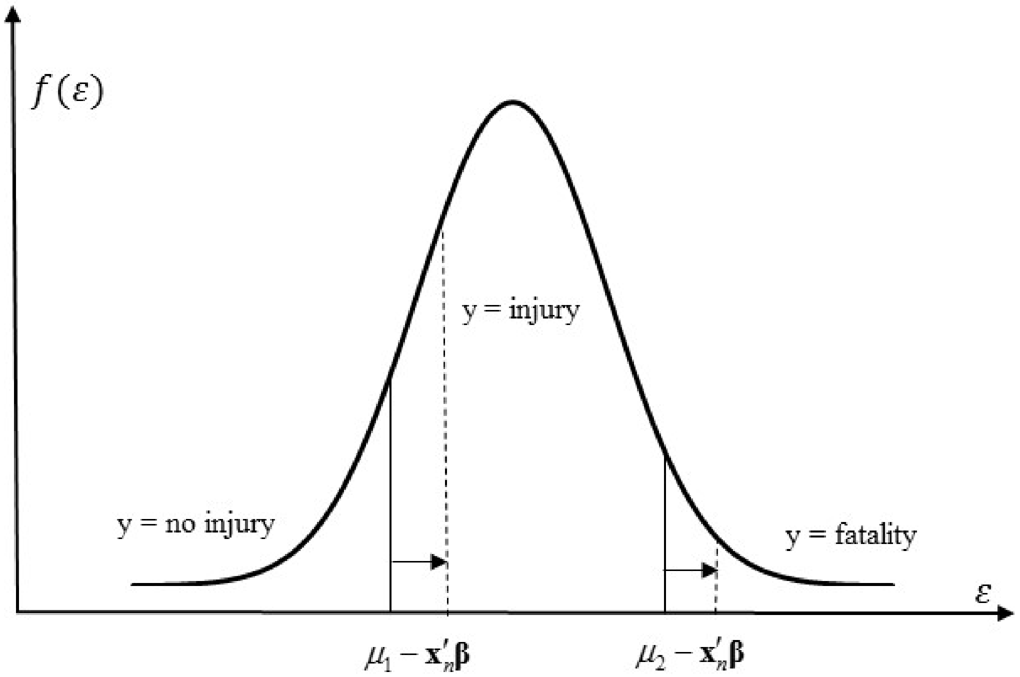
$$log\left(\frac{P_{ni}}{1-P_{ni}}\right) = \boldsymbol{x}_{\boldsymbol{n}}'\boldsymbol{\beta} - \mu_i \quad \text{i=1, …, I-1} \qquad (4.13)$$

Proportional Odds Assumption

- The fact that you can calculate odds ratios highlights a key assumption of ordered logit:
  - "Proportional odds assumption"
  - Also known as the "parallel regression assumption"
    - Which also applies to ordered probit
- Model assumes that variable effects on the odds of lower vs. higher outcomes are consistent; or regression parameters have to be the same for different response outcomes.
- If this assumption doesn't seem reasonable, consider multinomial logit, generalized ordered logistic and proportional odds model.

Airbag Example for Proportional Odds

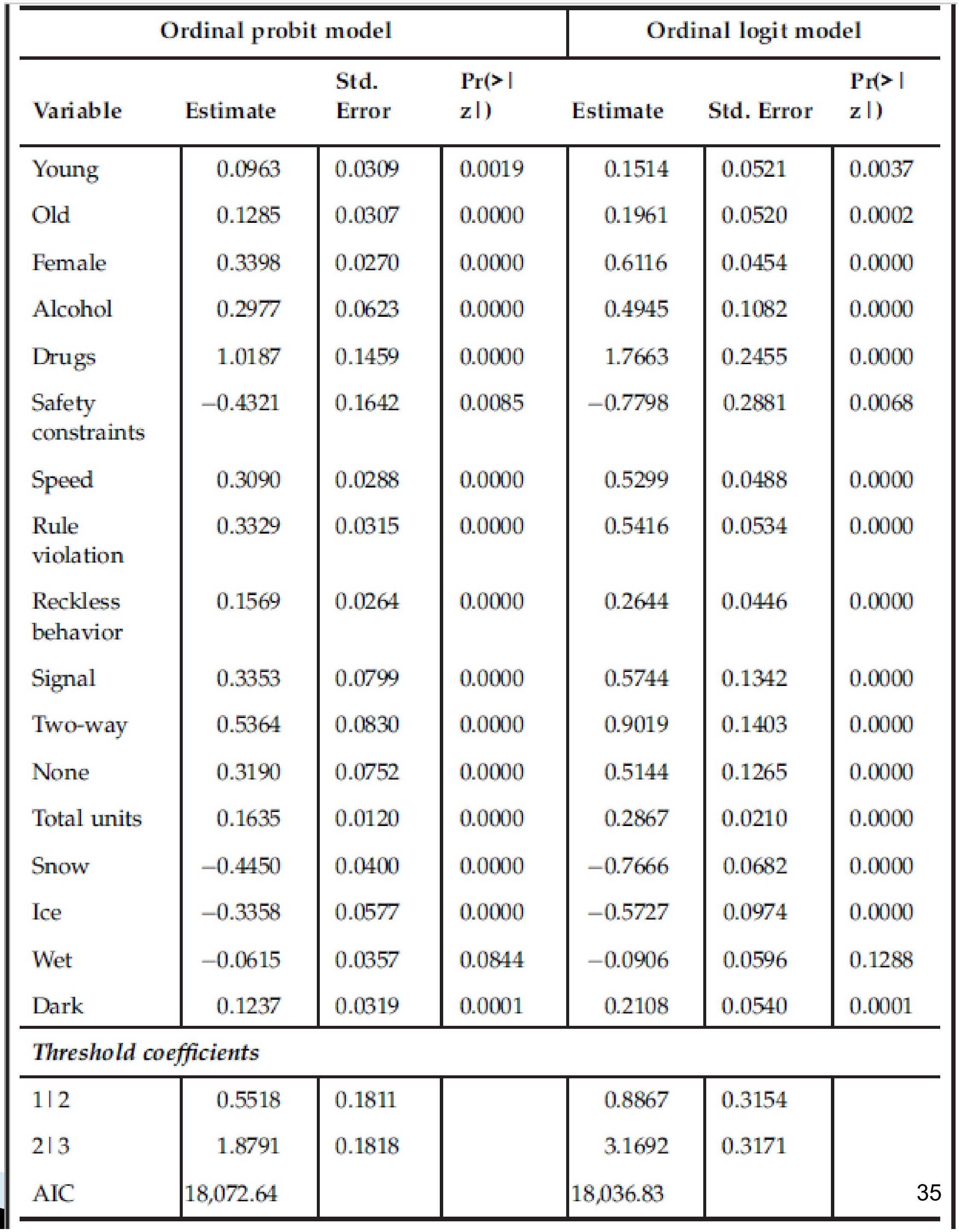
Consider a model of three injury levels - no injury, injury, and fatality. Suppose that one of the factors is airbag. A negative parameter of the airbag indicator (1 if it was deployed and zero otherwise) becomes greater and hence, shifts values to the right on the X-axle. Thus, the model constrains the effect of the seatbelt to simultaneously decrease the probability of a fatality and increase the no injury probability. But we know for a fact that the activation of an airbag may cause injury and/or decrease no injury; but unfortunately, ordered models cannot account for this bi-directional possibility because the shift in thresholds is constrained to move in the same direction.


$f(\varepsilon)$
y = injury
y = no injury
y = fatality
ε
$\mu_1 - \mathbf{x}_n'\boldsymbol{\beta}$
$\mu_2 - \mathbf{x}_n'\boldsymbol{\beta}$

OP Model Coefficient Estimates

Exercise 4.4: Coefficient Estimates for OP and OL*

*: partial results shown here



	Ordinal probit model			Ordinal logit model		
Variable	Estimate	Std. Error	Pr(>|z|)	Estimate	Std. Error	Pr(>|z|)
Young	0.0963	0.0309	0.0019	0.1514	0.0521	0.0037
Old	0.1285	0.0307	0.0000	0.1961	0.0520	0.0002
Female	0.3398	0.0270	0.0000	0.6116	0.0454	0.0000
Alcohol	0.2977	0.0623	0.0000	0.4945	0.1082	0.0000
Drugs	1.0187	0.1459	0.0000	1.7663	0.2455	0.0000
Safety constraints	−0.4321	0.1642	0.0085	−0.7798	0.2881	0.0068
Speed	0.3090	0.0288	0.0000	0.5299	0.0488	0.0000
Rule violation	0.3329	0.0315	0.0000	0.5416	0.0534	0.0000
Reckless behavior	0.1569	0.0264	0.0000	0.2644	0.0446	0.0000
Signal	0.3353	0.0799	0.0000	0.5744	0.1342	0.0000
Two-way	0.5364	0.0830	0.0000	0.9019	0.1403	0.0000
None	0.3190	0.0752	0.0000	0.5144	0.1265	0.0000
Total units	0.1635	0.0120	0.0000	0.2867	0.0210	0.0000
Snow	−0.4450	0.0400	0.0000	−0.7666	0.0682	0.0000
Ice	−0.3358	0.0577	0.0000	−0.5727	0.0974	0.0000
Wet	−0.0615	0.0357	0.0844	−0.0906	0.0596	0.1288
Dark	0.1237	0.0319	0.0001	0.2108	0.0540	0.0001
Threshold coefficients						
1|2	0.5518	0.1811		0.8867	0.3154	
2|3	1.8791	0.1818		3.1692	0.3171	
AIC	18,072.64			18,036.83		

Solution

- This exercise uses the same dataset as Exercise 4.1. In this exercise, an ordinal probit and an ordinal logistic regression model are respectively applied in order to recognize the ordinality of injury level, the dependent variable.

- First, determine the functional form: Eq. 4.12 for the ordinal probit model and Eq. 4.15 for the ordinal logistic model. In both equations, the μs are estimable thresholds, along with the parameter vector β.

- Second, estimate the coefficients using the R “ordinal” package:

```
crash_data_ordinal <- data_model_ch5
op_model <- clm(as.factor(INJSVR) ~ YOUNG + OLD + FEMALE + ALCFLAG + DRUGFLAG + SAFETY + 
DRVRPC_SPD + DRVRPC_RULEVIO + DRVRPC_RECK
          + TRFCONT_SIGNAL + TRFCONT_2WAY + TRFCONT_NONE
          + TOTUNIT + ROADCOND_SNOW + ROADCOND_ICE + ROADCOND_WET + LGTCOND_DARK,
          data = crash_data_ordinal, link = "probit")
```

- Note that the response (INJSVR) should be a factor, which will be interpreted as an ordinal response with levels ordered as in the factor. Replace “probit” with “logit” if you want to run an ordinal logit model. Other distribution options are: "cloglog", "loglog", "cauchit", "Aranda-Ordaz", "log-gamma".

- Third, present the model results of the coefficients and finally, summarize the findings.

Generalized Ordered Logistic and Proportional Odds Model

- A generalized ordered logistic model (gologit) provides results similar to those that result from running a series of binary logistic regressions/ cumulative logit models.
- The ordered logit model is a special case of the gologit model where the coefficients β are the same for each category.
- A gologit model and an MNL model, whose variables are freed from the proportional odds constraint, both generate many more parameters than an ordered logit model.
- The partial proportional odds model (PPO) is in between, as some of the coefficients β are the same for all categories and others may differ.
- A PPO model allows for the parallel lines/ proportional odds assumption to be relaxed for those variables that violate the assumption.

Generalized Ordered Logistic Model

In the gologit model, the probability of crash injury for a given crash can be specified as (I-1) set of equations:

$$Pr(y_n > i) = \frac{exp(\mathbf{x}_n^{'}\boldsymbol{\beta}_{\boldsymbol{i}} - \mu_i)}{1 + exp(\mathbf{x}_n^{'}\boldsymbol{\beta}_{\boldsymbol{i}} - \mu_i)}, i = 1, \dots (I - 1) \qquad (4.16)$$

Where μ_i is the cut-off point for the ith cumulative logit. Note that Equation 4.16 is different from Equation 4.14 in that $\boldsymbol{\beta}_{\boldsymbol{i}}$ is a single set of coefficients that vary by category *i*.

Partial Proportional Odds Model

In the PPO model formulation, it is assumed that some explanatory variables may satisfy the proportional odds assumption while some may not. The cumulative probabilities in the PPO model are calculated as follows:

$$Pr(y_n > i) = \frac{exp\left(\boldsymbol{x}'_n\boldsymbol{\beta}+\boldsymbol{T}'_n\boldsymbol{\gamma}_{\boldsymbol{i}}-\mu_i\right)}{1+exp\left(\boldsymbol{x}'_n\boldsymbol{\beta}+\boldsymbol{T}'_n\boldsymbol{\gamma}_{\boldsymbol{i}}-\mu_i\right)}, i = 1, \ldots (I-1) \qquad (4.17)$$

Where x_n is a (p×1) vector of independent variables of crash n, β is a vector of regression coefficients, and each independent variable has a β coefficient. T_n is a (q×1) vector (q≤p) containing the values of crash n on the subset of p explanatory variables for which the proportional odds assumption is not assumed, and γ_i is a (q×1) vector of regression coefficients. So, γ_i represents deviation from the proportionality β_i and is an increment associated only with the ith cumulative logit, i=1,⋯,(I-1).

Partial Proportional Odds Model (Cont'd)

An alternative but simplified way to think about the PPO model is to have two sets of explanatory variables: x$_1$, the coefficients of which remain the same for all injury severities and x$_2$, the coefficients of which vary across injury severities.

$$\Pr\left(y_n > i\right) = \frac{\exp\left(\mathbf{x}_1'\boldsymbol{\beta}_1 + \mathbf{x}_2'\boldsymbol{\beta}_{2i} - \mu_i\right)}{1 + \exp\left(\mathbf{x}_1'\boldsymbol{\beta}_1 + \mathbf{x}_2'\boldsymbol{\beta}_{2i} - \mu_i\right)} \qquad (4.18)$$

Sequential Logistic/Probit Regression Model

- Although the generalized ordered logit model relaxes the proportional odds assumption by allowing some or all of the parameters to vary by severity levels, the set of explanatory variables is invariant over all severity levels.
- The sequential logit/probit regression model should be considered when the difference in the set of explanatory variables at each severity level is important.
- Sequential logit/probit regression allows different regression parameters for different severity levels. A sequential logit/probit model supposes (I-1) latent variables given as (I-1) sets of equations.
- Sequential logistic regression not only accounts for the inherent order of dependent variables but also allows **different sets of regression parameters** to be independently considered in the model specification.

Sequential logistic/probit regression model

$$
\begin{aligned}
z_{n1} &= \boldsymbol{\alpha}_1 + \mathbf{x}'_n \boldsymbol{\beta}_1 + \varepsilon_{n1} \\
z_{n2} &= \boldsymbol{\alpha}_2 + \mathbf{x}'_n \boldsymbol{\beta}_2 + \varepsilon_{n2} \\
&\vdots \\
z_{n,I-1} &= \boldsymbol{\alpha}_{I-1} + \mathbf{x}'_n \boldsymbol{\beta}_{I-1} + \varepsilon_{n,I-1}
\end{aligned} \qquad (4.19)
$$

where z_{ni} is a continuous latent variable that determines whether the injury severity is observed as i or higher, β_i's are the vectors of estimated parameters, and ε_{ni}'s are error terms that are independent of x_n.

Sequential logistic/probit regression model structure

- The sequential model is a type of hierarchical model where lower stages mean lower injury severity.
- For example, stage 1 of the KABCO scale may be KABC versus O; stage 2 may be KAB versus C and stage 3 may be KA versus B. This change in definition matters when explaining the model results. Moreover, the hierarchical structure can be arranged from low to high or from high to low, which can also be called “forward” or “backward.”
- It is important to know that the sequential model uses a subpopulation of the data to estimate the variant set of β_i. The subpopulation decreases as the stages progresses forward or backward. In the forward format, all data are used in the first stage to estimate β_1, but only the crashes with injury type C or higher are used in the second stage to estimate β_2. Crashes with injury type B or higher are used in the second stage to estimate β_3.

Sequential logistic/probit regression model applications

- Jung et al. (2010) applied the sequential logit model to assess the effects of rainfall on the severity of single-vehicle crashes on Wisconsin interstate highways.
  - The sequential logit regression model outperformed the ordinal logit regression model in predicting crash severity levels in rainy weather when comparing goodness of fit, parameter significance, and prediction accuracies.
  - The sequential logit model identified that stronger rainfall intensity significantly increases the likelihood of fatal and incapacitating injury crash severity, while this was not captured in the ordered logit model.
- Yamamoto et al. (2008) also reported superior performance and unbiased parameter estimates with sequential binary models as compared with traditional ordered probit models, even when underreporting was a concern.

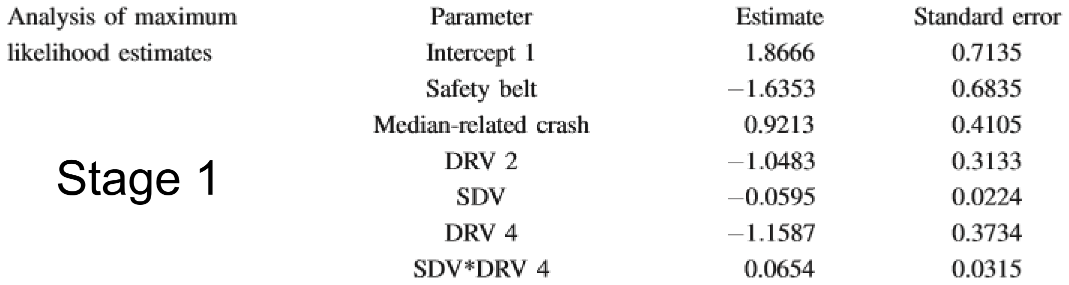
Reference: Soyoung Jung, Xiao Qin, David Noyce (2012). Injury Severity of Multi-Vehicle Crash in Rainy Weather, ASCE, Journal of Transportation Engineering, 138(1), pp. 1-12.

Forward format:

Stage 1: $(1-P)/P1 = \mathrm{EXP}(\alpha_1 + \beta X1) = h1$

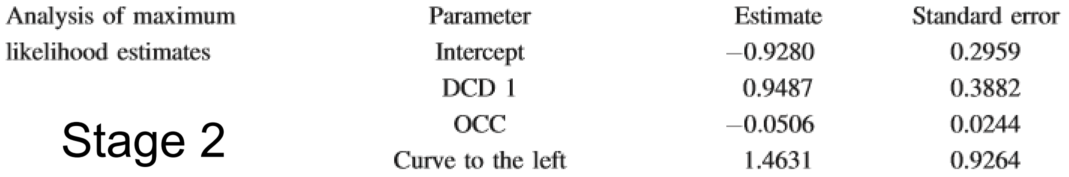
Stage 2: $P3/P2 = \mathrm{EXP}(\alpha_2 + \beta X2) = h2$

Stage 1



Analysis of maximum likelihood estimates	Parameter	Estimate	Standard error
	Intercept 1	1.8666	0.7135
	Safety belt	−1.6353	0.6835
	Median-related crash	0.9213	0.4105
	DRV 2	−1.0483	0.3133
	SDV	−0.0595	0.0224
	DRV 4	−1.1587	0.3734
	SDV*DRV 4	0.0654	0.0315

Stage 2



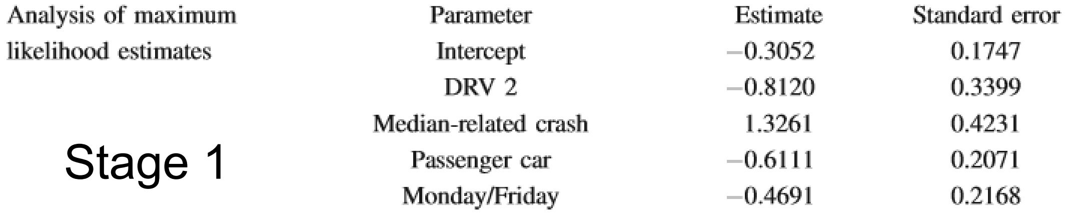
Analysis of maximum likelihood estimates	Parameter	Estimate	Standard error
	Intercept	−0.9280	0.2959
	DCD 1	0.9487	0.3882
	OCC	−0.0506	0.0244
	Curve to the left	1.4631	0.9264

Backward format:

Stage 1: $P3/(1-P3) = \mathrm{EXP}(\alpha_1 + \beta X1) = I1$

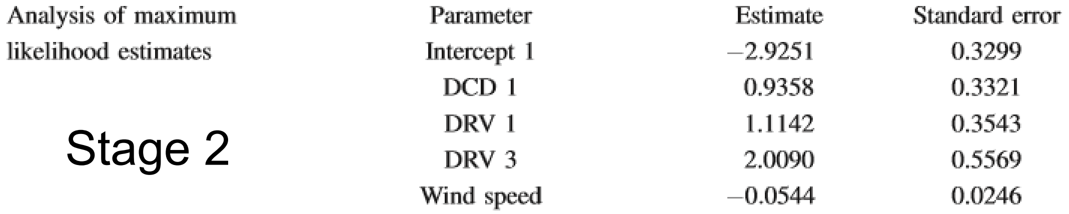
Stage 2: $P2/P1 = \mathrm{EXP}(\alpha_2 + \beta X2) = I2$

Stage 1



Analysis of maximum likelihood estimates	Parameter	Estimate	Standard error
	Intercept	−0.3052	0.1747
	DRV 2	−0.8120	0.3399
	Median-related crash	1.3261	0.4231
	Passenger car	−0.6111	0.2071
	Monday/Friday	−0.4691	0.2168

Stage 2



Analysis of maximum likelihood estimates	Parameter	Estimate	Standard error
	Intercept 1	−2.9251	0.3299
	DCD 1	0.9358	0.3321
	DRV 1	1.1142	0.3543
	DRV 3	2.0090	0.5569
	Wind speed	−0.0544	0.0246

Where
P1=probability of PDO;
P2=probability of possible injury; and
P3 = probability of fatal/incapacitating/non-incapacitating injury

DCD is defined as the minimum safe stopping distance (SSD)
OCC: Average 5-min OCC (%)

Model Interpretation

- To properly interpret model results, we need to be wary of the data formats as they can be structured differently because of different methods.
- The dependent variable can be treated as individual categories, categories higher than level i, or categories lower than level i.
- Independent variables can be continuous, indicator (1 or 0) or categorical.
- Categorical variables should be converted to dummy variables, with a dummy variable assigned to each distinct value of the original categories.
- The coefficient of a dummy variable can be interpreted as the log-odds for that particular value of dummy minus the log-odds for the base value which is 0 (e.g., the odds of being injured when drinking and driving is 10 times of someone who is sober).

Model Interpretation (Cont'd)

- The key concepts of marginal effect and elasticity are fundamental to understanding model estimates. The marginal effect is the unit-level change in y for a single-unit increase in x if x is a continuous variable.
- In a simple linear regression, the regression coefficient of x is the marginal effect, $\frac{\partial y}{\partial x_k} = \beta_k$.
- Due to the nonlinear feature of logit models, the marginal effect of any continuous independent variable is: $\frac{\partial p_i}{\partial x_{ki}} = \beta_{ki} p_{\ i} \left(1 - p_i\right)$.
- Such marginal effects are called instantaneous rates of change because they are computed for a variable while holding all other variables as constant.

Model Interpretation (Cont'd)

- Elasticity can be used to measure the magnitude of the impact of specific variables on the injury-outcome probabilities.
- For a continuous variable, elasticity is the % change in y given a 1% increase in x. It is computed from the partial derivative with respect to the continuous variable of each observation n.
- For indicator or dummy variables (those variables taking on values of 0 or 1), a pseudo elasticity of an indicator variable with respect to an injury severity category represents the percent change in the probability of that injury severity category when the variable is changed from zero to one.

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